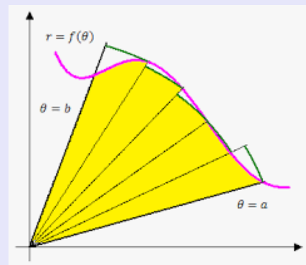
Calculus II

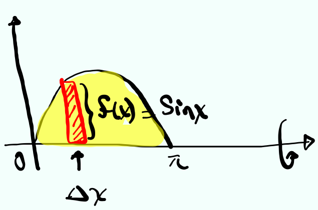
Lecture 9



Feb 19-8:47 AM

Class QZ 8 (open Notes)

Find the volume by rotating the region bounded by $\boxed{y = \sin x}$, $y=0$, $x=0$, $x=\pi$ about $\boxed{x\text{-axis}}$. Drawing required.



Disk Method

$$V = \int_0^{\pi} \pi [\sin x]^2 dx$$

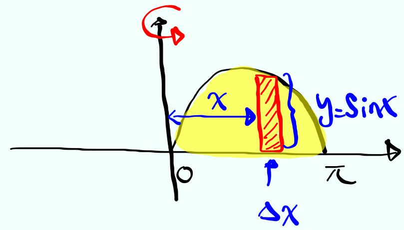
$$= \pi \cdot \int_0^{\pi} \sin^2 x\, dx =$$

$$= \pi \cdot \int_0^{\pi} \frac{1 - \cos 2x}{2} dx$$

$$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]\Big|_0^{\pi} = \boxed{\frac{\pi^2}{2}}$$

Jun 24-10:54 AM

Find the volume if we rotate the same region by y-axis.



Shell

$$V=\int_0^{\pi} 2\pi x \sin x \, dx$$

$$=2\pi\int_0^{\pi} x \sin x \, dx$$

$\int x \sin x \, dx$

$u=x \qquad dv=\sin x \, dx$

$du=dx \qquad v=-\cos x$

$=-x\cos x-\int -\cos x \, dx$

$=-x\cos x+\sin x$

$$=2\pi\Big[-x\cos x+\sin x\Big]\Big|_0^{\pi}$$

$$=2\pi\Big[-\pi\underbrace{\cos\pi}_{-1}+\underbrace{\sin\pi}_{0}+\underbrace{0\cdot\cos 0}_{0}-\underbrace{\sin 0}_{0}\Big]$$

$$=2\pi(\pi)=\boxed{2\pi^2}$$

Jun 25-8:06 AM

Evaluate $\int \tan^2 x \sec x \, dx$

$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1)\sec x \, dx$

$= \int \sec^3 x \, dx - \int \sec x \, dx$

$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C$

$= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$

$\int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx \qquad u=\sec x \qquad dv=\sec^2 x \, dx$

$du=\sec x \tan x \, dx \qquad v=\tan x$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \tan x \boxed{-\int \sec^3 x \, dx} + \int \sec x \, dx$

✓ $2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$

$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$

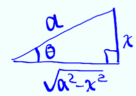
Jun 25-8:13 AM

Integration with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

we do these with trig. subs.

$\sqrt{a^2-x^2}$, $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sin\theta = \frac{x}{a}$



$$\sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$
$$= a\sqrt{\cos^2\theta} = a\cos\theta$$

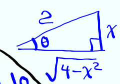
$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

$x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$

$dx = 2\cos\theta\,d\theta$

$\theta = \sin^{-1}\frac{x}{2}$



$$= \int \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta} \cdot 2\cos\theta\,d\theta$$

$$= \int \frac{2\cdot\cos\theta}{4\sin^2\theta} \cdot 2\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$1 + \cot^2\theta = \csc^2\theta$

$$= \int \cot^2\theta\,d\theta =$$

$$= \int (\csc^2\theta - 1)\,d\theta = \int \csc^2\theta\,d\theta - \int 1\,d\theta$$

$$= -\cot\theta - \theta + C$$

$$\boxed{= -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\frac{x}{2} + C}$$

Jun 25-8:25 AM

Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$

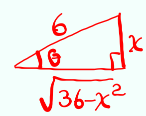
$a^2 = 36$

$a = 6$

$x = 6\sin\theta$

$\sin\theta = \frac{x}{6}$

$dx = 6\cos\theta\,d\theta$



$$\int \frac{x}{\sqrt{36-x^2}}\,dx = \int \frac{6\sin\theta}{\sqrt{36-36\sin^2\theta}} \cdot 6\cos\theta\,d\theta$$

$$= \int \frac{6\sin\theta}{\cancel{6\cos\theta}} \cdot \cancel{6\cos\theta}\,dx$$

$$= 6\int \sin\theta\,d\theta = -6\cos\theta$$

$$= -\cancel{6}\cdot\frac{\sqrt{36-x^2}}{\cancel{6}}\Big|_0^3$$

$$= -\left[\sqrt{36-9} - \sqrt{36-0}\right] = -3\sqrt{3} + 6$$

$$= \boxed{6 - 3\sqrt{3}}$$

Jun 25-8:39 AM

$x^2+y^2=R^2$

Circle

Center $(0,0)$

Radius R

Area $=\pi R^2$

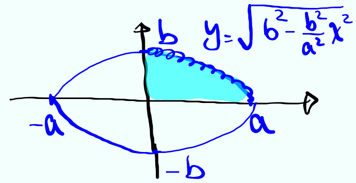
Circumference $=2\pi R$

Arc length

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Ellipse

Center $(0,0)$



Area

Circumference

Arc length

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$b^2x^2+a^2y^2=a^2b^2$$

$$a^2y^2=a^2b^2-b^2x^2$$

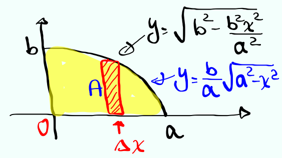
$$y^2=b^2-\frac{b^2}{a^2}x^2 \Rightarrow y=\sqrt{b^2-\frac{b^2}{a^2}x^2}$$

Jun 25-8:46 AM

$$\sqrt{b^2-\frac{b^2x^2}{a^2}}$$

$$=\sqrt{\frac{a^2b^2-b^2x^2}{a^2}}$$

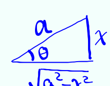
$$=\sqrt{\frac{b^2(a^2-x^2)}{a^2}}=\frac{b}{a}\sqrt{a^2-x^2}$$



$$A=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

$x=a\sin\theta$

$dx=a\cos\theta\,d\theta$



$$=\frac{b}{a}\int\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$$

$$=\frac{b}{\cancel{a}}\cdot\cancel{a^2}\int_0^{\pi/2}\cos^2\theta\,d\theta=ab\left[\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta\right]$$

$$=\frac{ab}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]\Bigg|_0^{\pi/2}=\frac{ab}{2}\left[\frac{\pi}{2}+0\right]$$

$$=\frac{ab\pi}{4}$$

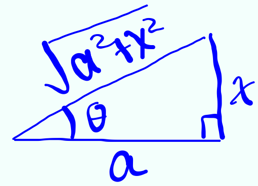
Area of the ellipse

$$A=4\left(\frac{\pi ab}{4}\right)\Rightarrow\boxed{A=\pi ab}$$

Jun 25-8:52 AM

what about $\sqrt{a^2+x^2}$?

$x = a \tan\theta$



$\tan\theta = \frac{x}{a}$ $\;\rightarrow\; -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$a^2 + x^2 =$

$a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta)$

$= a^2(\sec^2\theta)$ $\qquad \sqrt{a^2+x^2} = a\sec\theta$

Jun 25-9:05 AM

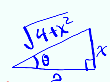
Evaluate $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

$a^2 = 4$

$a = 2$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$



$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$= \int \frac{8\tan^3\theta}{\sqrt{4+4\tan^2\theta}} \cdot 2\sec^2\theta\, d\theta$

$= \int \frac{16\tan^3\theta\sec^2\theta}{2\sec\theta}\, d\theta = 8\int \tan^3\theta \sec\theta\, d\theta$

$= 8\int \tan^2\theta \cdot \sec\theta\tan\theta\, d\theta$ $\qquad u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta = 8\int (u^2-1)\,du$

$= 8\left[\frac{u^3}{3} - u\right] + C$

$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$

$= \frac{\cancel{8}}{3}\left[\frac{\sqrt{4+x^2}}{\cancel{2}}\right]^3 - \overset{4}{\cancel{8}} \cdot \frac{\sqrt{4+x^2}}{\cancel{2}} + C$

$\boxed{= \frac{1}{3}\sqrt{(4+x^2)^3} - 4\sqrt{4+x^2} + C}$

$= \frac{1}{3}\cdot(4+x^2)\sqrt{4+x^2} - 4\sqrt{4+x^2} + C$

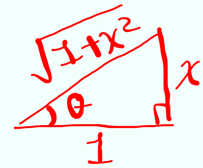
Jun 25-9:09 AM

Evaluate $\int_0^1 \sqrt{1+x^2}\,dx$

$a=1$

$x = a\tan\theta$

$x = \tan\theta$



$dx = \sec^2\theta\, d\theta$

$$I = \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta = \boxed{\text{See earlier notes}}\Big|_0^{\pi/4} = \boxed{}$$

Jun 25-9:22 AM

$\sqrt{x^2-a^2}$

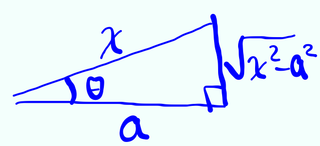
$x = a\sec\theta$

$0 \le \theta < \frac{\pi}{2}$ or

$\pi \le \theta < \frac{3\pi}{2}$

$\sec\theta = \frac{x}{a}$

$\cos\theta = \frac{a}{x}$



$dx = a\sec\theta\tan\theta\, d\theta$

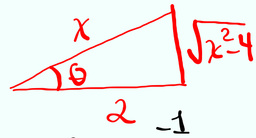
Jun 25-9:53 AM

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

$a^2=4$

$a=2$

$x=2\sec\theta$



$\sec\theta=\frac{x}{2}$ $\rightarrow$ $\theta=\sec^{-1}\frac{x}{2}$

$\cos\theta=\frac{2}{x}$

$dx=2\sec\theta\tan\theta\,d\theta$

$$=\int\frac{\sqrt{4\sec^2\theta-4}}{\cancel{2\sec\theta}}\cdot\cancel{2\sec\theta}\tan\theta\,d\theta$$

$$=\int 2\tan\theta\tan\theta\,d\theta$$

$$=2\int\tan^2\theta\,d\theta=2\int(\sec^2\theta-1)\,d\theta$$

$$=2(\tan\theta-\theta)+C$$

$$\boxed{=2\left[\frac{\sqrt{x^2-4}}{2}-\sec^{-1}\frac{x}{2}\right]+C}$$

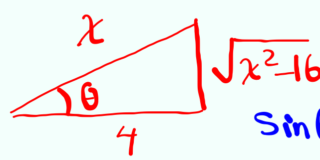
Jun 25-9:55 AM

$$\int\frac{dx}{x^2\sqrt{x^2-16}}$$

$a^2=16$

$a=4$

$x=4\sec\theta$



$\sin\theta=\frac{\sqrt{x^2-16}}{x}$

$dx=4\sec\theta\tan\theta\,d\theta$

$$=\int\frac{4\sec\theta\tan\theta\,d\theta}{16\sec^2\theta\cdot\sqrt{16\sec^2\theta-16}}$$

$$=\int\frac{\cancel{4}\,\cancel{\sec}\theta\,\cancel{\tan\theta}}{16\sec^{\cancel{2}}\theta\cdot\cancel{4}\cdot\cancel{\tan\theta}}\,d\theta=\frac{1}{16}\int\frac{1}{\sec\theta}\,d\theta$$

$$=\frac{1}{16}\int\cos\theta\,d\theta=\frac{1}{16}\sin\theta+C=\frac{1}{16}\cdot\frac{\sqrt{x^2-16}}{x}+C$$

Jun 25-10:02 AM

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$$

$9x^2 = (3x)^2$

$u = 3x \longrightarrow \frac{u}{3} = x$

$du = 3dx$

$\frac{du}{3} = dx$

$x = \frac{\sqrt{2}}{3} \to u = \sqrt{2}$

$x = \frac{2}{3} \to u = 2$

$$= \int_{\sqrt{2}}^{2} \frac{\frac{du}{3}}{\left(\frac{u}{3}\right)^5\sqrt{u^2-1}}$$

$$= \int_{\sqrt{2}}^{2} \frac{\frac{1}{3}}{\frac{u^5}{243}\sqrt{u^2-1}}\,du$$

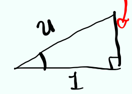
$$= \frac{243}{3}\int_{\sqrt{2}}^{2} \frac{1}{u^5\sqrt{u^2-1}}\,du$$

$$= 81\int_{\sqrt{2}}^{2} \frac{1}{u^5\sqrt{u^2-1}}\,du$$

$u = \sec\theta$

$\cos\theta = \frac{1}{u}$

$\sqrt{u^2-1}$



$du = \sec\theta\,\tan\theta\,d\theta$

$$= 81\int_{\pi/4}^{\pi/3} \frac{\cancel{\sec\theta}\ \cancel{\tan\theta}}{\sec^{\cancel{5}}\theta\cdot\cancel{\tan\theta}}\,d\theta = 81\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta$$

$$= 81\int_{\pi/4}^{\pi/3}\left(\cos^2\theta\right)^2 d\theta = 81\int_{\pi/4}^{\pi/3}\left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta$$

Jun 25-10:08 AM

$$\int\sqrt{x^2+2x}\,dx$$

Hint: Use completing the square.

$x^2+2x = x^2+2x+1-1$

$= (x+1)^2-1$

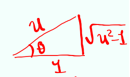
$$= \int\sqrt{(x+1)^2-1}\,dx$$

$u = x+1 \qquad du = dx$

$$= \int\sqrt{u^2-1}\,du$$

$u = \sec\theta$

$\cos\theta = \frac{1}{u}$



$du = \sec\theta\tan\theta\,d\theta$

$$= \int\sqrt{\sec^2\theta-1}\cdot\sec\theta\tan\theta\,d\theta$$

$$= \int\tan^2\theta\cdot\sec\theta\,d\theta = \int(\sec^2\theta-1)\sec\theta\,d\theta$$

$$= \int\sec^3\theta\,d\theta - \int\sec\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\ln\left|u+\sqrt{u^2-1}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x} - \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x}\right| + C$$

Jun 25-10:32 AM

Using partial Fractions

$$\frac{x}{x^2+x-2} = \frac{x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$x = A(x-1) + B(x+2) \Rightarrow \begin{cases} A+B=1 \\ 2B-A=0 \end{cases}$$

$$x = (A+B)x + 2B - A$$

$$B = \frac{1}{3}$$

$$A = \frac{2}{3}$$

$$\frac{x}{x^2+x-2} = \frac{\frac{2}{3}}{x+2} + \frac{\frac{1}{3}}{x-1}$$

Partial Fractions

$$\int \frac{x}{x^2+x-2}dx = \int \frac{\frac{2}{3}}{x+2}dx + \int \frac{\frac{1}{3}}{x-1}dx$$

$$= \frac{2}{3}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{x-1}dx$$

$$= \frac{2}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

$$= \frac{1}{3}\ln\left|\left[(x+2)^2\cdot(x-1)\right]\right| + C$$

Jun 25-10:43 AM

$$\int_0^1 \frac{x-4}{x^2-5x+6}dx \qquad \frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\int_0^1 \frac{x-4}{x^2-5x+6}dx = \qquad x-4 = A(x-3)+B(x-2)$$

$$x=3 \qquad -1 = A\cdot 0 + B\cdot 1$$

$$\boxed{B=-1}$$

$$\int_0^1 \left[\frac{2}{x-2} - \frac{1}{x-3}\right]dx$$

$$x=2 \qquad -2 = A\cdot(-1) + B\cdot 0$$

$$\boxed{A=2}$$

$$= \left(2\ln|x-2| - \ln|x-3|\right)\Big|_0^1$$

$$= \ln\left|\frac{(x-2)^2}{x-3}\right|\Bigg|_0^1 = \ln\frac{1}{2} - \ln\frac{4}{3}$$

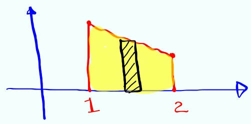
$$= \ln\frac{\frac{1}{2}}{\frac{4}{3}} = \boxed{\ln\frac{3}{8}}$$

Jun 25-10:52 AM

Find the area below $f(x)=\frac{1}{x^3+x}$, above x-axis, from $x=1$ to $x=2$.
Drawing required.

$f(1)=\frac{1}{2}$, $f(2)=\frac{1}{10}$

$f(x)\geq 0$ on $[1,2]$



$$A=\int_1^2 \frac{1}{x^3+x}\,dx$$

$$\frac{1}{x^3+x}=\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$1=A(x^2+1)+(Bx+C)x$$

$$1=Ax^2+A+Bx^2+Cx$$

$$1=(A+B)x^2+Cx+A$$

$A+B=0 \to B=-1$

$C=0$

$A=1$

$$\int_1^2 \frac{1}{x^3+x}\,dx=$$

Error x

$$\int_1^2\left(\frac{1}{x}-\frac{1}{x^2+1}\right)dx$$

$$=\Big[\ln x-\tan^{-1}x\Big]_1^2$$

$$=\ln 2-\tan^{-1}2-\cancel{\ln 1}^{\to 0}+\cancel{\tan^{-1}1}^{\to \frac{\pi}{4}}$$

$$=\boxed{\ln 2-\tan^{-1}2+\frac{\pi}{4}}$$

Jun 25-11:00 AM

$$=\frac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\cos^2 2\theta\right]d\theta$$

$$=\frac{81}{4}\left[\theta+\cancel{2}\cdot\frac{\sin 2\theta}{\cancel{2}}\right]\Bigg|_{\pi/4}^{\pi/3}+\frac{81}{4}\int_{\pi/4}^{\pi/3}\cos^2 2\theta\,d\theta$$

$$=\frac{81}{4}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{2}-\frac{\pi}{4}-1\right]+\frac{81}{4}\int_{\pi/4}^{\pi/3}\frac{1+\cos 4\theta}{2}\,d\theta$$

$$=\frac{81}{4}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{2}-\frac{\pi}{4}-1\right]+\frac{81}{8}\left[\theta+\frac{\sin 4\theta}{4}\right]\Bigg|_{\pi/4}^{\pi/3}$$

$$=\frac{81}{4}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{2}-\frac{\pi}{4}-1\right]+\frac{81}{8}\left[\frac{\pi}{3}-\frac{\sqrt{3}}{8}-\frac{\pi}{4}-0\right]$$

$\frac{162}{8}$

$$=\frac{1}{8}\left[\cancel{\frac{162\pi}{3}}+\cancel{\frac{162\sqrt{3}}{2}}-\cancel{\frac{162\pi}{4}}-162+\cancel{\frac{81\pi}{3}}-\cancel{\frac{81\sqrt{3}}{8}}-\cancel{\frac{81\pi}{4}}\right]$$

$$=\frac{1}{8}\left[\frac{243\pi}{3}+\frac{567\sqrt{3}}{8}-\frac{243\pi}{4}-162\right]$$

$$=\boxed{}\checkmark$$

Jun 25-10:20 AM